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# Stability and Pattern of Bifurcating Steady States in Systems with Double Involution

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A system of ordinary differential equations is said to be a reversible system if there exists an involution such that the vector field is reversed under this involution. The steady-state systems of reaction–diffusion systems (RD-systems) on 1-dimensional space are all reversible ODE systems provided the kinetic terms do not involve the derivatives of component functions. If, in addition, a RD-system exhibits $Z(2)$-symmetry, then the $Z(2)$-symmetry introduces an additional involution to its steady-state system. We are interested in how this additional symmetry property affects the temporal stability and pattern of the bifurcating steady states in such RD-systems. We study the class of RD-systems whose steady-state systems are reversible with respect to two linear involutions whose product is a reversal symmetry. The set of fixed points of these involutions are both 2-dimensional planes in the 4-dimensional state space. They intersect at the constant steady states where bifurcating steady states emanate from. The steady states being studied are periodic solutions bifurcated from a constant steady state at 1 : 1 resonance in the steady-state system. Their temporal stability is analyzed through a center manifold reduction on the RD-system. A complete description of their spatial structure is also obtained. Two models, the Lambda–Omega system and a predator–prey model, are studied as examples. We conclude that all small-amplitude bifurcating steady states are temporally stable in the Lambda–Omega system and unstable in the predator–prey model. Numerical approximations of bifurcating steady states in these two models, obtained by solving both the RD-system and the steady-state system, are presented as numerical verification of our analysis. We also conclude that the bifurcation regime near 1 : 1 resonance in the steady-state system of the predator–prey model is hyperbolic. 

## 1. INTRODUCTION

For many years reaction–diffusion systems have been widely used in many fields to model phenomena involving interaction and diffusion pro-



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cesses Among all the interesting analysis, bifurcations of stationary waves has become a major concern in recent years especially in the study of biological sciences. Typically, this phenomenon occurs when two events take place simultaneously, namely, a Hopf bifurcation at a constant steady state in the steady-state system and a loss of temporal stability of the constant steady state [10]. Whether or not the bifurcating steady states are temporally stable is a nontrivial question. Most of the existing stability results are obtained through a linear stability analysis, which is closely related to bifurcation. However, for small-amplitude solutions, it is also possible to analyze their temporal stability by calculating the bifurcation equation. For example, the Brussellator [2].

In this paper, we are concerned with RD-systems of the following form,

$$\begin{aligned} u_t &= d_1 u_{xx} + f_1(u, v, k), \\ v_t &= d_2 v_{xx} + f_2(u, v, k), \end{aligned} \tag{1.1}$$

$$x \in [0, l],\ k \in R^m, \qquad u_x(0) = v_x(0) = u_x(l) = v_x(l) = 0.$$

We assume certain conditions on the kinetic terms to assure the existence of nontrivial steady states [13, 15]. These are small-amplitude steady states obtained through diffusive instability. That is, if one varies the diffusive rates across certain critical values, these steady states will bifurcate from a constant steady state under a Hopf bifurcation in the steady-state system.

The temporal stability of the small-amplitude bifurcating steady states are studied through center manifold reduction. Our attempt is to show that under certain conditions, the linear operator of the RD-system at a constant steady state has a 1-dimensional kernel. When reduced to the center manifold, the bifurcation of steady states is pitchfork. The pitchfork bifurcation is caused by $O(2)$-symmetry which is obtained by imbedding Neumann boundary conditions to an RD-system on $[-l, l]$ with periodic boundary conditions [3]. The temporal stability of the bifurcating steady states is thus a trivial consequence if the constant steady state loses temporal stability simultaneously as the pitchfork bifurcation occurs. Their pattern is also studied. Because their trajectories are symmetric with respect to the set of fixed points of the involutions in the state space, which is a consequence of the reversibility, these solutions can be expressed in a simple oscillatory form.

In summary, the bifurcating steady states are temporally stable solutions under a pitchfork bifurcation if the constant steady state loses stability at the critical value. We also show that if the steady-state system of

(1.1) is reversible under two involutions, the spatial structure of the bifurcating steady states at 1 : 1 resonance can be determined explicitly. This is illustrated with numerical evidence with two examples. Because of such explicit representation of symmetric cycles, the bifurcation regime in the steady-state systems can be determined by computing certain plane curves. We demonstrate this analysis with the ecological model in section 6.

## 2. Reversible Systems with Double Involution

Definition 2.1. A 2n-dimensional system

$$\dot{U} = F(\mu, U), \qquad \mu \in R^m, F(\mu, 0) = 0 \tag{2.1}$$

is a reversible system if there exists an involution $I: R^{2n} \to R^{2n}$, $I^2$ is the identity, such that $I(0) = 0$, $(I\dot{U}) = -F(\mu, IU)$. Here "$\cdot$" denotes "$d/dx$".

When the involution $I$ is linear, we say (2.1) is reversible with respect to $I$ of type $(n, n)$ if the characteristic polynomial of the linear operator $I$ has the form $(\lambda + 1)^n(\lambda - 1)^n$. Let Fix $I$ be the set of fixed points of the linear transformation $I$, then dim(Fix $I$) $= n$ and $0 \in$ Fix $I$ [5, 1, 9].

Reversible systems and Hamiltonian systems share many similar properties. A theory analogous to the Lyapunov center theorem has been proven by Devaney for the reversible systems [5]. We state this theorem below in the setting suitable for our discussion.

Theorem 2.1 (Lyapunov–Devaney Center Theorem). *Suppose* (2.1) *is a reversible system of type* $(n, n)$ *whose Jacobian matrix has simple purely imaginary eigenvalues* $\pm i\omega (\omega > 0)$ *and other eigenvalues* $\pm\lambda_2$, $\pm\lambda_3, \ldots, \pm\lambda_n$. *If none of the ratios* $\lambda_i/i\omega$, $2 \le i \le n$, *is an integer, then there exists a smooth one-parameter family of periodic solutions, each symmetric with respect to* Fix $I$, *in a neighborhood of* 0. *This family can be parametrized by a variable, with the periods and amplitudes of periodic solutions varying smoothly with this variable.*

Lemma 2.1. *All coupled harmonic oscillators of the following form*

$$\begin{aligned} u_{xx} &= f_1(\mu, u, v, u_x, v_x), \\ v_{xx} &= f_2(\mu, u, v, u_x, v_x), \end{aligned} \tag{2.2}$$

*are reversible systems with respect to the linear involution*

$$I_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

*if $f_1, f_2$ are both even functions in $u_x$ and $v_x$. Here* Fix $I_1 = \{(u, u_x, v, v_x) \mid u_x = v_x = 0\} = u$, *$v$-plane.*

The reversibility plays an important role in determining the bifurcation regimes in reversible systems and in determining the structure of bifurcating periodic solutions. The entire state space of (2.2) is symmetric with respect to Fix $I_1$. If a periodic orbit penetrates Fix $I_1$ in the 4-dimensional state space, then its trajectory must be symmetric with respect to Fix $I_1$. In this case, we call this periodic solution a "symmetric cycle." If, on the other hand, a periodic orbit does not penetrate Fix $I_1$, then it must be paired by another periodic solution symmetrically across Fix $I_1$. In this case, we call this pair of periodic solutions a "symmetric pair." Mathematical expressions of such solutions are stated below.

DEFINITION 2.2. Denote the reflectional action on a periodic solution $\gamma(x) = \gamma(x + P)$ by $\rho : x \to P - x$; $\rho \circ \gamma(x) = \gamma(P - x)$. We say $\gamma(x)$ is symmetric with respect to Fix $I_1$ if $\gamma(x) = I_1(\rho \circ \gamma(x))$.

THEOREM 2.2. *If $\gamma(x) = \gamma(x + P)$ is a periodic solution of* (2.1), *so is $I_1(\rho \circ \gamma(x))$. Moreover, if the trajectory of $\gamma(x)$ penetrates* Fix $I_1$ *in the* 4-*dimensional state space*, $\gamma(x) = I_1(\rho \circ \gamma(x))$.

COROLLARY 2.1. A periodic solution $\gamma(x)$ is called a "symmetric cycle" if $\gamma(x) = I_1(\rho \circ \gamma(x))$. If $\gamma(x) \neq I_1(\rho \circ \gamma(x))$, then $\gamma(x)$ and $I_1(\rho \circ \gamma(x))$ are called a "symmetric pair" of periodic solutions.

Let $\mu_1 = 1/d_1$, $\mu_2 = 1/d_2$ in (1.1). According to Lemma 2.1, the steady-state system of (1.1) is reversible under the involution $I_1 : (u, u_x, v, v_x) \to (u, -u_x, v, -v_x)$. Apparently, any constant steady state $Q_k$ lies on Fix $I_1$.

Now we shift the coordinates from the origin to $Q_k$. The steady-state system of (1.1) has the following form

$$\begin{aligned} \dot{u} &= u_x, \\ \dot{u}_x &= -\mu_1 g_1(u, v, k), \\ \dot{v} &= v_x, \\ \dot{v}_x &= -\mu_2 g_2(u, v, k). \end{aligned} \tag{2.3}$$

DEFINITION 2.3. An O.D.E. system $\dot{U} = F(\mu, U)$ is $Z(2)$-symmetric if it is invariant under the action of the two element group $\{\mathbf{I}, \mathcal{R}\}$, where $\mathbf{I}$ is the identity and $\mathcal{R}U \to -U$.

Assume that (2.3) is $Z(2)$-symmetric. This assumption implies that $g_1$, $g_2$ are odd functions in $u$ and $v$. That is, $g_i(-u, -v, k) = -g_i(u, v, k)$, $i = 1, 2$. The main reason for this assumption is because the $Z(2)$-symmetry renders more structure to the steady-state system. Nonetheless, it occurs naturally in many applications, especially when there are certain symmetries exhibited by the modelled phenomenon. The $Z(2)$-symmetry assumed here then introduces an additional involution $I_2 = -I_1$. Therefore, (2.3) is reversible under two involutions given by $I_1:(u, u_x, v, v_x) \to (u, -u_x, v, -v_x)$ and $I_2:(u, u_x, v, v_x) \to (-u, u_x, -v, v_x)$. Moreover, $\text{Fix } I_1 \cap \text{Fix } I_2 = \{(0, 0, 0, 0)\}$.

Since $\text{Fix } I_1 = u, v$-plane ($u_x = v_x = 0$) and $\text{Fix } I_2 = u_x, v_x$-plane ($u = v = 0$), it is clear that all constant steady states lie on Fix $I_1$. Therefore, when the steady-state system (2.3) undergoes a Hopf bifurcation, the bifurcating periodic solutions become steady states of the RD-system. All such bifurcating steady states are periodic in $x$ with each wave symmetric with respect to the vertical lines through its local extrema [14]. In the case when there exist two involutions $I_1$ and $I_2$, the trajectories of such solutions are then symmetric with respect to Fix $I_1$ as well as Fix $I_2$. The symmetry with respect to Fix $I_2$ implies that the waves of a periodic solution $(u(x), v(x))$ are symmetric with respect to the lines $u = v = 0$. That is, $(-u(x), -v(x)) = (u(x + P/2), v(x + P/2))$ where $P$ is the period. Because of the double symmetry, we can express the steady states emanated from $Q_k$ by

$$P_{ss}(x) = Q_k + \bar{r}\varepsilon(x) = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \varepsilon(x), \qquad r_1, r_2 \in R, \tag{2.4}$$

or simply by

$$P_{ss}(x) = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \varepsilon(x), \qquad r_1, r_2 \in R, \tag{2.5}$$

if the coordinates are shifted to $Q_k$. Here $\varepsilon(x)$ is a periodic function oscillating around 0 with symmetric waves. At 1 : 1 resonance, the Jacobian matrix of (2.2) has a pair of eigenvalues $\pm i\omega_0$ of multiplicity two. For small-amplitude solutions, $\varepsilon(x)$ has a period close to $2\pi/\omega_0$. We normalize its amplitude such that $|\varepsilon(x_c)| = 1$, where $x_c$ are the critical points of $\varepsilon(x)$, that is, $\varepsilon'(x_c) = 0$.

To illustrate the symmetry of such solutions in the phase space, we examine a numerical solution of the Lambda–Omega system (Fig. 3a). Projections of the trajectory of this solution onto $u$, $v$-plane and $u_x$, $v_x$-plane in the 4-dimensional phase space are shown in Figs. 3b and 3c, respectively. The two end points of the curve shown in Fig. 3b are where the trajectory penetrates the $u$, $v$-plane ($u_x = v_x = 0$). They are the local extrema of this solution. In Fig. 3c, the projected trajectory clearly exhibits the symmetry with respect to the origin, which is the $u$, $v$-plane (Fix $I_1$). For such solutions, these two projections must be straight lines with the origin in the middle of these segments. This is a necessary outcome from the double symmetry. Moreover, although these segments (projections) do not have to have the same length, they must have the same slope on the respective projection planes. In this particular example, they have the same slope as well as length. More details will be elaborated in Section 5.

## 3. Bifurcation at 1 : 1 Resonance through Diffusive Instability

We focus on a particular constant steady state of (1.1) and study the bifurcations of nonuniform steady states through diffusive instability. We use ($\mu_1$, $\mu_2$) in (2.1) as bifurcation parameters, with $k$ in the kinetic terms being constants. Since $d_1$, $d_2$ are positive real numbers, therefore, $0 < \mu_1, \mu_2 < \infty$.

From the theory of centers we know that, if the eigenvalues of the Jacobian matrix of a linear system are all complex conjugate pairs of purely imaginary numbers, then the solutions near the equilibrium are all periodic solutions. In general, adding nonlinear terms will cause topological metamorphosis in the phase space of the vector field and, therefore, the linearization technique is not applicable to the nonhyperbolic singular points of a nonlinear system. However, it has been proven that the topological structure in a neighborhood of a nonhyperbolic singular point is preserved if the nonlinearly perturbed system falls into either one of two classes of systems: Hamiltonian systems and reversible systems.

For the ODE system (2.2), the Devaney center theorem (Theorem 2.1), guarantees the existence of a family of periodic solutions whenever its Jacobian matrix has a pair of purely imaginary eigenvalues. By varying ($\mu_1$, $\mu_2$), the bifurcation of periodic solutions is thus an obvious consequence of the emergence of purely imaginary eigenvalues.

Suppose the Jacobian matrix of (2.2) expanded at a constant steady state $Q_k$ has eigenvalues $\pm i\omega_1$, $\pm i\omega_2$, ($\omega_1$, $\omega_2 > 0$). Then there exist two families of periodic steady states each emanate from $Q_k$ when the ratio

$\omega_1/\omega_2$ is not (a rational number) (nonresonance). When the ratio $\omega_1/\omega_2$ is an integer $n$, we say these two families interact at $1:n$ resonance. At certain resonances, these two families may merge into one.

Return to the PDE setting, when a spatial interval is prescribed, possible periodic steady states are limited to the ones satisfying the boundary conditions with frequency close to the real component of the purely imaginary eigenvalues. We derive, in the following, sufficient conditions for such bifurcations to occur through an analysis of the phase dynamics of (2.2).

Consider the steady-state system of (1.1) in the following form,

$$\begin{aligned} u_{xx} &= -\mu_1 f_1(u, v, k), \\ v_{xx} &= -\mu_2 f_2(u, v, k). \end{aligned} \tag{3.1}$$

Let $f_{11} = \partial f_1(Q_k)/\partial u$, $f_{12} = \partial f_1(Q_k)/\partial v$, $f_{21} = \partial f_2(Q_k)/\partial u$, $f_{22} = \partial f_2(Q_k)/\partial v$. The variational system of (3.1) at $Q_k$ has the following vector form with $z = (u, v)' - Q_k$,

$$z_{xx} = Lz, \qquad L = \begin{bmatrix} -\mu_1 f_{11} & -\mu_1 f_{12} \\ -\mu_2 f_{21} & -\mu_2 f_{22} \end{bmatrix}. \tag{3.2}$$

$L$ has a negative real eigenvalue $q$ of multiplicity two, if both of the following conditions hold,

(1) $\mu_1 f_{11} + \mu_2 f_{22} > 0$,

(2) $(\mu_1 f_{11} - \mu_2 f_{22})^2 + 4\mu_1\mu_2 f_{12} f_{21} = 0$.

Define the region $G = \{(\mu_1, \mu_2) \mid \mu_1 f_{11} + \mu_2 f_{22} > 0 \text{ and } (\mu_1 f_{11} - \mu_2 f_{22})^2 + 4\mu_1\mu_2 f_{12} f_{21} > 0\}$ on $\mu_1$, $\mu_2$-plane. If $f_{ij}$ meet certain restrictions [15], $G$ intersects the first quadrant of $\mu_1$, $\mu_2$-plane at a nonempty set and condition (2) corresponds to the boundary line $\Gamma_{11}$ of $G$. Such a region is illustrated in Fig. 1, there are two possible locations of $G$, depending on the sign of $f_{22}$. If $(\mu_1, \mu_2)$ lies on $\Gamma_{11}$, the Jacobian matrix of (2.1) at $Q_k$ has one pair of purely imaginary eigenvalues $\pm i\sqrt{-q}$ of multiplicity two, with $q = (-\mu_1 f_{11} - \mu_2 f_{22})/2$. When $(\mu_1, \mu_2)$ varies across $\Gamma_{11}$ entering $G$ (nonresonance), two families of periodic solutions emerge and detach from the equilibrium. Such bifurcation at $1:1$ resonance in reversible systems is very similar to the Hamiltonian Hopf bifurcation at $1:1$ resonance [11]. There is an important difference between such bifurcations and the classical Hopf bifurcation. That is, the periodic solutions are not isolated in the Hamiltonian or Reversible Hopf bifurcation. There is a family of periodic solutions for each parameter value.

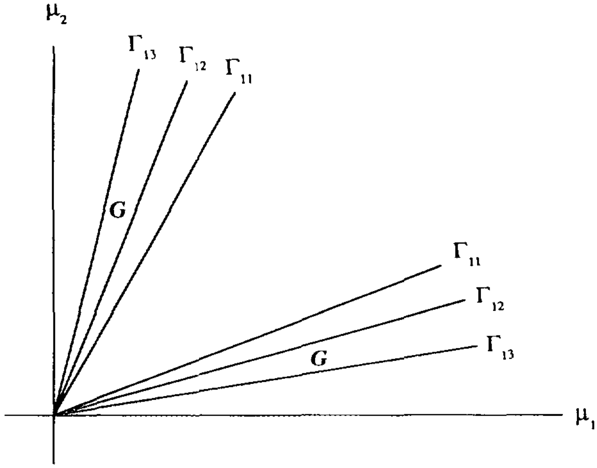


FIGURE 1

Under no-flux boundary conditions, the solutions are given by

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cos\sqrt{-q}x. \tag{3.3}$$

With the presence of nonlinear terms, since (3.1) is a reversible system, the periodic solutions near $Q_k$ will not be destroyed but merely nonlinearly distorted in the phase space. For small-amplitude solutions, their periods are close to $2\pi/\sqrt{-q}$. If, in addition, the expansion of (3.1) at $Q_k$ has the form of (2.2), then these periodic solutions can be expressed in the form of (2.4). The period of $\varepsilon(x)$ is thus close to $2\pi/\sqrt{-q}$.

## 4. STABILITY OF THE BIFURCATING STEADY STATES

The temporal stability of nonuniform steady states in an RD-system is closely related to the stability of the constant steady state where bifurcation occurs. Bifurcated steady states are stable in time, provided the constant steady state loses temporal stability as the bifurcation parameter crosses the critical value. But if the constant steady state is unstable in time for the parameter values in a full neighborhood of the critical value, then the bifurcated nonuniform steady states are also unstable in time. This is illustrated in Section 5 with two models. The constant steady state of the predator–prey model is temporally unstable in a full neighborhood of the critical bifurcation value. Therefore, bifurcating steady states are unstable in time. But the constant steady state in the Lambda–Omega system loses temporal stability when the bifurcation of steady states occurs. Therefore, the bifurcating steady states are stable in time.

In this section we study the temporal stability of the periodic steady states emanated from $Q_k$ in (3.1) using center manifold reduction. The effects of acting symmetry groups to the system are discussed. For simplicity, we rescale the interval length $[0, l]$ to $[0, \pi]$. This scales the diffusion coefficients by a factor of $(\pi/l)^2$. However, we do not incorporate the scaling factor explicitly in $\mu_1$, $\mu_2$, since it does not affect the result. Suppose the Taylor expansion of (4.1) at $Q_k$ is given by

$$\begin{aligned} u_t &= d_1 u_{xx} + g_1(u, v, k), \\ v_t &= d_2 v_{xx} + g_2(u, v, k), \end{aligned} \tag{4.1}$$

with $g_i(-u, -v, k) = -g_i(u, v, k)$, $i = 1, 2$. Then there exist two involutions $I_1 : (u, u_x, v, v_x) \to (u, -u_x, v, -v_x)$, and $I_2 : (u, u_x, v, v_x) \to (-u, u_x, -v, v_x)$ with $\text{Fix } I_1 \cap \text{Fix } I_2 = \{(0, 0, 0, 0)\}$. Let $\mu = (\mu_1, \mu_2)$, we denote by $\mu_c$ the critical value where $Q_k$ undergoes a stability changeover. If $Q_k$ retains stability for parameter in a neighborhood of the critical value, the bifurcating periodic steady states do not gain stability and numerical simulations will converge to $Q_k$. Let $f_{ij}$ be as defined in (3.2) and write (4.2) in vector form

$$U_t = LU + N(U), \tag{4.2}$$

where

$$L = \begin{bmatrix} d_1\Delta + f_{11} & f_{12} \\ f_{21} & d_2\Delta + f_{22} \end{bmatrix} \quad \text{and} \quad N(0) = N'(0) = 0.$$

We assume that $Q_k$ loses temporal stability when $\mu$ varies across $\mu_c$, and $L$ has a 1-dimensional kernel when $\mu = \mu_c$. Denote the adjoint of $L$ by $L^*$. Let $\phi$ be a nonzero vector in $\ker(L)$, $\psi$ be a nonzero vector in $\ker(L^*)$. That is, $L\phi = 0$, $L^*\psi = 0$. By Fredholm alternatives, $\ker(L^*) = (\text{range } L)^\perp$, one may normalize the vectors such that $\langle \phi, \psi \rangle = 1$. The space $\mathcal{L}^2(0, \pi)$ has the standard inner product

$$\langle u, v \rangle = \frac{2}{\pi} \int_0^\pi u(\theta) v(\theta)\, d\theta.$$

Let $X = \ker(L)$, $Y = \text{range}(L)$ and $Z = X \oplus Y$. To each $U \in Z$, one obtains the following decomposition,

$$U = \xi\phi + y,$$

with $\xi\phi = \langle U, \psi\rangle\phi \in X$, $y = U - \xi\phi \in Y$. Obviously, $\langle\psi, y\rangle = 0$. $X$ has dimension 1 and $Y$ has codimension 1. To distinguish from the notation of spatial derivative $u' = u_x$, we use "·" to denote the time derivative $\dot{u} = u_t$. Therefore,

$$U_t = \dot{\xi}\phi + \dot{y} = L(\xi\phi + y) + N(\xi\phi + y),$$

and

$$\begin{aligned}
\dot{\xi} &= \langle U_t, \psi\rangle, \\
&= \psi \cdot L(\xi\phi + y) + \psi \cdot N(\xi\phi + y), \\
&= \psi \cdot N(\xi\phi + y), \\
\dot{y} &= U_t - \dot{\xi}\phi, \\
&= Ly + (N - (\psi \cdot N)\phi).
\end{aligned}$$

Under this decomposition, (4.2) becomes

$$\begin{aligned}
\dot{\xi} &= \psi \cdot N, \\
\dot{y} &= Ly + (N - (\psi \cdot N)\phi).
\end{aligned} \tag{4.3}$$

For no-flux boundary conditions, the minimally degenerate singularity is always a pitchfork, regardless the existence of $Z(2)$-symmetry in (4.2) or not. This is because the Neumann boundary conditions can be imbedded into periodic boundary conditions which then introduces $O(2)$-symmetry for reaction–diffusion systems. That is, the Neumann boundary conditions is treated as a symmetry constraint on the problems with periodic boundary conditions [3, 6]. As a consequence, the reduced equation (projected vector field on the center manifold) is an odd polynomial in $\xi$. Computationally, this is reflected by the fact that the integration of $\cos^n(\kappa x)$ over $[0, \pi]$, for any nonzero integer $\kappa$, is zero when $n$ is odd. Therefore, all the even order terms in the reduced equation vanish.

We assume $\vartheta(\xi)$, with $\vartheta : (-\xi, \xi) \to Y$, be the center manifold that inherits this symmetry,

$$\vartheta(\xi) = \alpha_1\xi^3 + O(\xi^5). \tag{4.4}$$

Project the vector field onto the center manifold, one has the 1-dimensional equation,

$$\dot{\xi} = \beta_1 \xi^3 + O(\xi^5). \tag{4.5}$$

If $\beta_1 \neq 0$, with appropriate change of coordinates, the bifurcation is a pitchfork characterized by the normal form $(\mu - \mu_c)\xi + \beta_1 \xi^3$. The sign of $\beta_1$ determines whether the bifurcation is supercritical or subcritical, it also determines the stability ($\beta_1 < 0$) or instability ($\beta_1 > 0$) of the constant steady state $Q_k$ when $\mu = \mu_c$. The bifurcating solutions then have the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \xi\phi + O(\xi^3).$$

Their spatial structure is close to $\cos(\kappa x)$ when $\xi$ is small. Their temporal stability is also determined by the pitchfork bifurcation.

## 5. Applications and Numerical Results

Two models are studied in this section, the Lambda–Omega system and a predator–prey model. For each of these models, we first perform a pattern analysis to obtain certain information in the pattern of the steady states, then we discuss the temporal stability of the small-amplitude, bifurcating solutions using the analysis outlined in the previous section. Numerical results obtained by solving the RD-model when the steady states are stable in time, and by solving the steady-state system when the steady states are unstable in time, are also presented.

*Model* 1: Lambda–Omega System [8].

$$\begin{aligned} u_t &= d_1 u_{xx} + \lambda u - \omega v, \\ v_t &= d_2 v_{xx} + \omega u - \lambda v. \end{aligned} \tag{5.1}$$

This class of systems was introduced to illustrate the application of Hopf theory to the steady-state system in the study of plane-wave solutions. Here we study this system with $\lambda = 1 - u^2 - v^2$, $\omega = 2$ on 1-dimensional interval $[0, l]$ with no-flux boundary conditions. There are three constant steady states, $Q_k = (0, 0), (q, -q), (-q, q)$, with $q = \sqrt{\frac{3}{2}}$. At $Q_k = (0, 0)$, the system is reversible under the two involutions $I_1$, $I_2$, as stated in Section 2. The 1 : 1 resonance line on the parameter plane is described by

$\mu_2 = m_l\mu_1$, with $m = 7 - 4\sqrt{3}$. Under the two involutions, the small-amplitude periodic steady states originated from $Q_k$ have the form

$$u_{ss}(x) = r_1\varepsilon(x), \qquad v_{ss}(x) = r_2\varepsilon(x). \tag{5.2}$$

We examine the correspondence of local extrema and relative amplitudes between the $u$, $v$ components of these solutions. This is done by substituting (5.2) into the steady-state system and examining the sign of the following product at the critical points $x_c$,

$$\begin{aligned} u_{xx}v_{xx} &= \mu_1\mu_2[(r_1(1 - r_1^2 - r_2^2) - 2r_2][2r_1 - r_2(1 - r_1^2 - r_2^2)], \\ &= \mu_1\mu_2(r_1 - 2r_2)(2r_1 - r_2) + O(r^3). \end{aligned} \tag{5.3}$$

The sign of this product determines the relative concavity of these two components at $x_c$. When $r_1$, $r_2$ are small, the higher order terms are insignificant. It is thus easy to see that one must have $r_1r_2 > 0$. Because if $r_1r_2 < 0$, then $u$ attains its maximal value while $v$ having a minimum, and vice versa. But in this case, $u_{xx}v_{xx} > 0$, which means that these two components both attain a maximum or both reaching a minimum. A contradiction. Therefore, we conclude that $r_1r_2 > 0$, the maxima (resp. minima) of $u_{ss}(x)$ correspond to the maxima (resp. minima) of $v_{ss}(x)$. Consequently, $u_{xx}v_{xx} > 0$ implies that $(r_1 - 2r_2)(2r_1 - r_2) > 0$. We conclude in the following theorem the pattern of these bifurcating steady states.

THEOREM 5.1. *For small-amplitude periodic steady states bifurcated from $Q_k$ at 1 : 1 resonance, the two components $u$, $v$ are concave in the same direction at a critical point. Moreover, their amplitudes are governed by*

$$(r_1 - 2r_2)(2r_1 - r_2) > 0. \tag{5.4}$$

Now we study the temporal stability of the bifurcating steady states (5.2) using the analysis outlined in Section 4. First we perform a linear stability analysis of $Q_k = (0, 0)$. The variational system has the form of (4.3). The eigenvalues of 1-dimensional Laplacian on $[0, \pi]$ with no-flux boundary conditions are $\lambda_j = -j^2$, $j = 0, 1, 2, \ldots$. Restrict the linear operator $L$ of (5.1) onto the 2-dimensional subspace generated by $\cos(jx)$, one has

$$L_j = \begin{bmatrix} -d_1j^2 + 1 & -2 \\ 2 & -d_2j^2 - 1 \end{bmatrix}.$$

The temporal stability of $Q_k$ is determined by the eigenvalues satisfying the characteristic equation:

$$\Lambda^2 + K_1\Lambda + K_2 = 0, \tag{5.5}$$

where

$$K_1 = (d_1 + d_2)j^2,$$
$$K_2 = j^4 + (\mu_2 - \mu_1)j^2 + 3\mu_1\mu_2.$$

$Q_k$ is unstable in time if either $K_1 < 0$ or $K_2 < 0$ for some $j$. Here we point out that $K_1$ and $K_2$ cannot vanish simultaneously. This guarantees that there is at most one zero eigenvalue (temporal) in each eigenspace. Since $K_1 > 0$, the stability of $Q_k$ is therefore determined by the sign of $K_2 = \det(L_j)$, $j = 1, 2, \ldots$. Define

$$h(x) = x^2 - (\mu_2 - \mu_1)x + 3\mu_1\mu_2, \tag{5.6}$$

then $\det(L_j) = h(\lambda_j)$. The function $h(x)$ is positive for all $x$ if $(\mu_1, \mu_2) \notin G$. The graph of $h(x)$ is a parabola tangent to the $x$-axis when $(\mu_1, \mu_2) \in \Gamma_{11}$ and it has two $x$-intercepts if $(\mu_1, \mu_2) \in G$ (Fig. 2). At $\mu_c = (\mu_{1c}, \mu_{2c})$, the equation $h(x) = 0$ has an integer solution $-\kappa^2$, where $\kappa$ is a positive

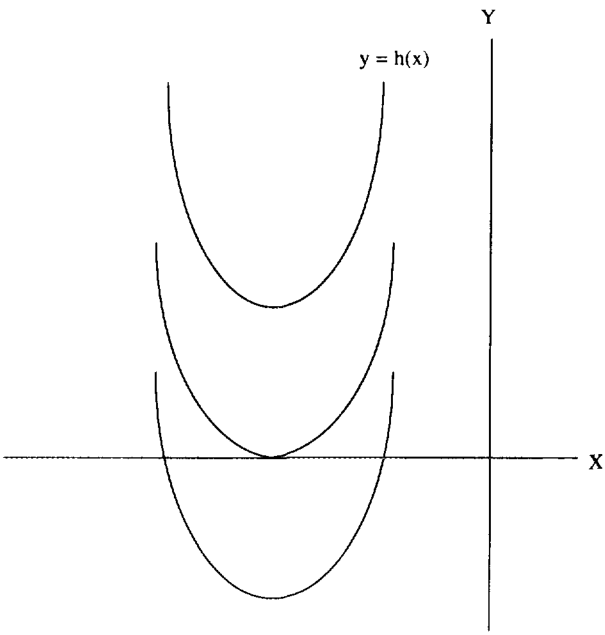


FIGURE 2

integer. That is,

$$\kappa^4 + (\mu_{2c} - \mu_{1c})\kappa^2 + 3\mu_{1c}\mu_{2c} = 0. \tag{5.7}$$

Recall our definition in (2.1) that $\mu = (\mu_1, \mu_2) = (1/d_1, 1/d_2)$, and $\mu_c = (1/d_{1c}, 1/d_{2c})$ is where $Q_k$ undergoes a stability changeover. There are a couple of ways to simplify the notation of bifurcation parameters. Since the bifurcation is determined mostly by the ratio of the diffusion rates, one may rescale the model by redefining $d_1 = 1$ and $d = d_2/d_1$, then study the bifurcations induced by varying the single parameter $d$. However, here we choose to define $\mu < \mu_c$ if $\mu_1/\mu_2 < \mu_{1c}/\mu_{2c}$ and $\mu > \mu_c$ if $\mu_1/\mu_2 > \mu_{1c}/\mu_{2c}$. Precisely, this means $Q_k$ is stable in time for $\mu < \mu_c$ and unstable for $\mu > \mu_c$. When $\mu = \mu_c$, $L$ has a 1-dimensional kernel $X$ and all other eigenvalues of $L$ are either purely imaginary or having negative real part.

Let $\phi$ be the normalized eigenvector of $L$ associated to the zero eigenvalue, then

$$\phi = (a_1 \cos \kappa x,\ a_2 \cos \kappa x)',$$

$$a_1 = \frac{2}{\sqrt{(1 - d_{1c}\kappa^2)^2 + 4}} \qquad a_2 = \frac{(1 - d_{1c}\kappa^2)a_1}{2}.$$

Therefore, $X = \ker(L) = \xi\phi$, with $\xi(t)$, $t \in R$. Let $L^*$ be the adjoint of $L$, and let $Y = \text{range}(L)$. Denote by $\psi$ the eigenvector of $L^*$ associated to the zero eigenvalue. By Fredholm alternative, $(\text{range } L)^\perp = \ker L^*$, thus for each $\eta \in Y$, the inner product $\langle \psi, \eta \rangle = 0$. The eigenvector $\psi$ can be determined uniquely by requiring the normalization $\langle \phi, \psi \rangle = 1$. We have

$$\psi = (b_1 \cos \kappa x,\ b_2 \cos \kappa x)',$$

$$b_1 = \frac{d_{2c}\kappa^2 + 1}{(d_{1c} + d_{2c})\kappa^2 a_1},\ b_2 = \frac{-2b_1}{d_{2c}\kappa^2 + 1}.$$

All the other eigenspaces of $L$ are spanned by eigenvectors of the form $\bar{c} \cos jx$ where $\bar{c} \in R^2$, $j = 0, 1, 2, \dots, j \neq \kappa$.

Let $Z = X \oplus Y$, $X$ has dimension 1 and $Y$ has codimension 1. For each $U = (u, v)'$, $U \in Z$, it has the following expansion

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} y_{1j} \\ y_{2j} \end{pmatrix} \cos(jx) e^{L_j t},$$

where $(y_{1j}, y_{2j})'$ are constant vectors. A decomposition is given by $U = \xi\phi - \eta$, with $\xi = \langle U, \psi \rangle$ and $\eta = U - \xi\phi \in Y$.

The validity of our analysis relies on the fact that ker($L$) is 1-dimensional. This is generically true. There are parameter values $(\mu_1, \mu_2)$ such that dim(ker $L$) = 2. But this set has measure zero on the parameter plane. To be specific, we examine the zeros of $h(x)$ which determine the unstable modes. First we look at some special bifurcation values. These are parameter values such that $\mu_c \in \Gamma_{11}$. That is, the parabola described by $h(x)$ in (5.6) is tangent to the $x$-axis at an integer $-\kappa^2$. All such parameter values can be determined explicitly. Let $(\mu_{1c}, \mu_{2c}) \in \Gamma_{11}$ such that $h(x)$ has a zero of multiplicity two. From (5.6), this implies that

$$\kappa^2 = \frac{\mu_{1c} - \mu_{2c}}{2}, \tag{5.8}$$

for some positive integer $\kappa$. But we also know $\mu_{2c} = (7 - 4\sqrt{3})\mu_{1c}$ at $1:1$ resonance. Substitute this relation into (5.8), one concludes that $\sqrt{(2\sqrt{3} - 3)\mu_{1c}}$ must be a positive integer. Therefore, $\mu_{1c} = n^2/(2\sqrt{3} - 3)$, $n = 1, 2, \ldots$. In other words, the parabola of $h(x)$ is tangent to the $x$-axis at an integer if and only if $(\mu_1, \mu_2) = (n^2/(2\sqrt{3} - 3), (2\sqrt{3} - 3)n^2/3)$ for $n = 1, 2, \ldots$ in (5.6). If this is the case then there is a "spontaneous bifurcation". That is, the bifurcation of $P_{ss}(x)$ at $1:1$ resonance in the steady-state system and the stability changeover of $Q_k$ (which induces the pitchfork bifurcation in the RD-system), occur simultaneously.

When $\mu_c \notin \Gamma_{11}$, there are two possibilities. If $h(x)$ has a unique integer solution $-\kappa^2$, then ker($L$) is 1-dimensional. Our analysis is valid for parameter values in a neighborhood of $\mu_c$ before subsequent bifurcations occur. In other words, for parameter values in this neighborhood, there is only one integer $-\kappa^2$ satisfying the inequality $h(x) \le 0$. On the other hand, if the parabola described by $h(x)$ intersects the $x$-axis at two integers $-\kappa^2$ and $-(\kappa + 1)^2$, then our analysis fails. In this case, ker($L$) is a 2-dimensional space spanned by $\bar{c} \cos \kappa x$ and $\bar{d} \cos(\kappa + 1)x$ for some constant vectors $\bar{c}, \bar{d} \in R^2$. We do not consider the case of codimension 2 bifurcation in this paper.

We assume that $h(x)$ has a unique integer zero. Define the projection $P : Z \to X$ as

$$PU = \langle \psi, U \rangle.$$

Under this decomposition, $U_t = \dot{\xi}\phi + \dot{\eta}$, and we have

$$\begin{aligned} \dot{\xi} &= PN(\xi\phi + \eta), \\ \dot{\eta} &= (I - P)L\eta + (I - P)N(\xi\phi + \eta). \end{aligned} \tag{5.9}$$

Denote the center manifold by $\vartheta(\xi) = \alpha_1\xi^3 + O(\xi^5)$. Restrict the vector field onto $\vartheta(\xi)$. The temporal stability of $Q_k$ when $\mu = \mu_c$ is determined by the 1-dimensional equation

$$
\begin{aligned}
\dot{\xi} &= PN(\xi\phi + \vartheta(\xi)), \\
&= \tfrac{3}{4}[b_2(a_1^2 a_2 + a_2^3) - b_1(a_1 a_2^2 + a_1^3)]\xi^3 + O(\xi^5), \\
&= \tfrac{3}{4}(a_2 b_2 - a_1 b_1)\xi^3 + O(\xi^5).
\end{aligned}
$$

But according to (5.7),

$$
\begin{aligned}
a_2 b_2 - a_1 b_1 &= a_1 b_1 (d_{1c}\kappa^2 - d_{2c}\kappa^2 - 2)/(d_{2c}\kappa^2 + 1), \\
&= -(d_{1c} d_{2c}\kappa^4 + 5)/(d_{1c}\kappa^2 + d_{2c}\kappa^2) < 0.
\end{aligned}
$$

Therefore, it is clear that if we define $\beta_1(\mu) = \frac{3}{4}(a_2 b_2 - a_1 b_1)$, then $\beta_1(\mu) < 0$, $\forall \mu$. We thus have a pitchfork bifurcation characterized by the bifurcation equation

$$
\dot{\xi} = (\mu - \mu_c)\xi + \beta_1(\mu)\xi^3. \tag{5.10}
$$

We summarize our stability analysis for the bifurcating periodic steady states in Lambda–Omega system in the following theorem.

THEOREM 5.2. *When $\mu$ increases across $\mu_c$, the constant steady state $Q_k = (0, 0)$ loses temporal stability. The system undergoes a pitchfork bifurcation at $Q_k$ characterized by* (5.10). *All the bifurcating periodic steady states are temporally stable with spatial structure close to the eigenfunction $\phi$ when $\mu$ is close to $\mu_c$.*

Numerical approximation of a small-amplitude steady-state solution bifurcated from $Q_k = (0, 0)$ when $(\mu_1, \mu_2) = (1/(2\sqrt{3} - 3), (2\sqrt{3} - 3)/3)$ is plotted in Fig. 3a. This solution is obtained by solving the reaction-diffusion system as an initial-boundary value problem over $[0, 2\pi]$ and letting $t \to \infty$. Numerical integrations with various initial states all converge to and stabilized at this solution after finite time. Many other steady states with different amplitudes and wave lengths are obtained similarly. This provides a numerical verification of our conclusion in Theorem 5.2. According to the numerical data, the amplitudes of the $u$, $v$ components of this solution are $r_1 \approx 1.8675E - 4$, $r_2 \approx 5.004E - 5$, and the period is $2\pi$. The two components of this solution concave in the same direction at the critical points, as described in our pattern analysis. Their amplitudes also satisfy the inequality (5.4) given in Theorem 5.1, with $(r_1 - 2r_2)(2r_1 - r_2) \approx 2.8034E - 8 > 0$. Moreover, according to Theorem 5.2, the spatial structure of this solution is close to the eigenfunction $\phi$. This is also true. By (5.8), $\kappa^2 = 1$. Thus $\phi$ has a period of $2\pi$, same as the numerical solution. To compare the amplitudes, we need to take into our consideration the scaling factor of diffusion rates and that $\phi$ is normalized. These factors combine into the factor $\delta \approx 3.73203$. That is, $\binom{r_1}{r_2} \approx \delta\binom{a_1}{a_2}$. Calculating $a_1$, $a_2$, we have $\phi = \binom{0.96593}{0.25882}\cos(x)$. We therefore find, with great precision, $\binom{r_1}{r_2}\cos(x) \approx \delta\phi$.

Demonstrated in Figs. 3b and 3c are projections of this solution in the 4-dimensional phase space onto $u$, $v$-plane and $u_x$, $v_x$-plane, respectively. We now explain why these projections are straight lines having the same slope on their respective projection planes. We know that the bifurcating solutions have a spatial structure close to the eigenfunction $\phi$ when $\mu$ is close to $\mu_c$. Therefore, they are expressed as $\binom{u}{v} = \binom{r_1}{r_2}\cos(\kappa x)$, if the unstable mode is spanned by $\bar{c}\cos(\kappa x)$. The projection on $u_x$, $v_x$-plane is obtained by plotting $\binom{u_x}{v_x} = \binom{-r_1\kappa}{-r_2\kappa}\sin(\kappa x)$ with $x$ varying. It should be easy for one to see that these segments of projections must be straight lines with the same slope $r_2/r_1$ on their respective planes. In this particular example, the unstable mode is generated by $\phi = \binom{a_1}{a_2}\cos(x)$. Therefore, these projections have same length and the segments in Figs. 3b and 3c appear identical.

*Model* 2: A Predator–Prey Model.

$$\begin{aligned} u_t &= d_1 u_{xx} + a_1 u + u \cdot z_1(u, v), \\ v_t &= d_2 v_{xx} + a_2 v + v \cdot z_2(u, v). \end{aligned} \tag{5.11}$$

Where $z_1(u, v) = b_1 u^2 + c_1 uv$, $z_2(u, v) = b_2 v^2 + c_2 uv$, $b_1$, $b_2$, $c_1$ are negative constants and $c_2$ is positive. This model arises in mathematical ecology and describes the evolution of population densities of a prey and a predator species ($u$ and $v$, respectively). Bifurcation and stability of positive solutions in this model with homogeneous Dirichlet boundary conditions have been studied [4, 12]. Here we study this model with homogeneous Neumann boundary conditions. The coefficients $a_1$ and $a_2$ represent, respectively, the birth rate of $u$ and $v$, which are assumed to be positive numbers. We also assume that the growth rate of each species is controlled in the absence of the other species. The functions describing the interactions between $u$ and $v$ are assumed to satisfy the following criterion:

(1) $z_1(u, v)$, $z_2(u, v)$ are $C^1$-functions of $u$, $v$ over $[0, l]$, $l > 0$.

(2) $z_1(0, 0) = z_2(0, 0) = 0$ and $(\partial z_1/\partial v)(u, v) < 0$, $(\partial z_2/\partial u)(u, v) > 0$ on $[0, l]$.

(3) $z_1(u, 0)$, $z_2(0, v)$ are strictly monotone decreasing in $u \geq 0$, $v \geq 0$ and

$$\lim_{u\to\infty} z_1(u, 0) = -\infty, \lim_{v\to\infty} z_2(u, v) = -\infty, \qquad \forall u \geq 0.$$

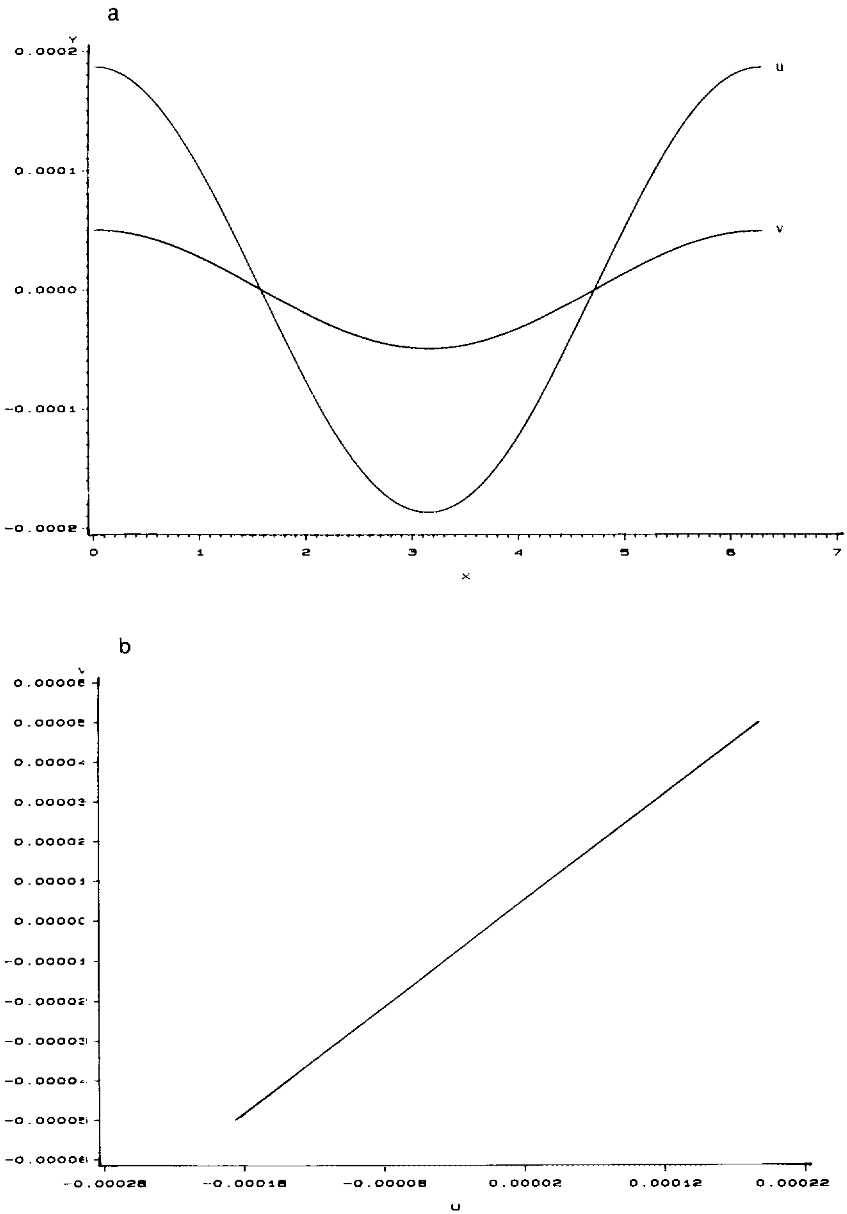
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FIGURE 3

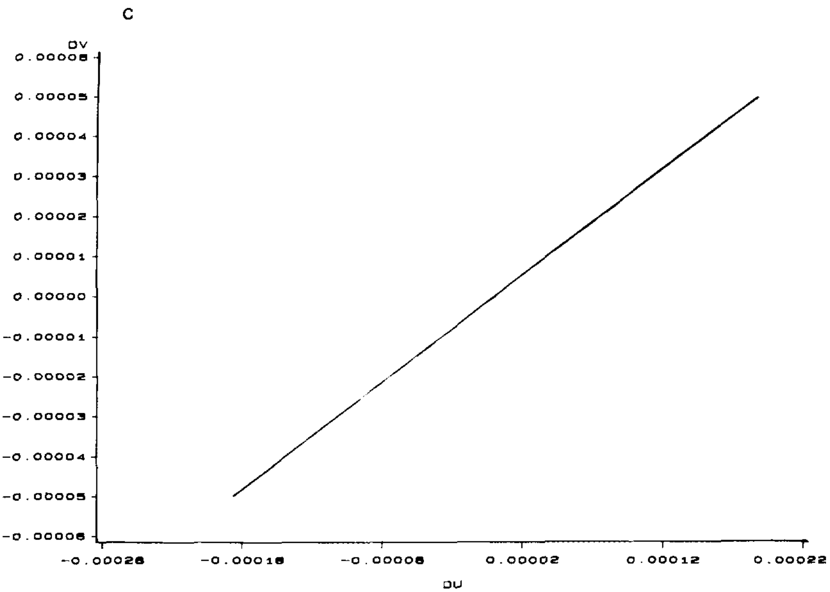



FIG. 3—*Continued*

It is easy to see, from a linear stability analysis, that this model has an unstable constant steady state $Q_k = (0, 0)$. The steady-state system at $Q_k$ is reversible under the two involutions $I_1, I_2$ stated in Section 2. Its Jacobian matrix has two pairs of purely imaginary eigenvalues given by $\pm\sqrt{\mu_1 a_1}i$ and $\pm\sqrt{\mu_2 a_2}i$. For each pair of eigenvalues, there is a family of periodic solutions emanating from $Q_k$ with their periods close to $2\pi/\sqrt{\mu_1 a_1}$ and $2\pi/\sqrt{\mu_2 a_2}$, respectively. As $(\mu_1, \mu_2)$ varies on the parameter plane, these four eigenvalues vary along the imaginary axis. When $\mu_2 = (a_1/a_2)\mu_1$ (the 1 : 1 resonance line on the parameter plane), these two families of periodic solutions interact at 1 : 1 resonance.

Since $Q_k$ is unstable in time for parameter values $\mu$ in a full neighborhood of the critical bifurcation value $\mu_2 = (a_1/a_2)\mu_1$, the bifurcating periodic steady states are unstable in time, according to the argument stated in Section 4. Indeed, numerical integrations of (5.11) forward in time with various initial states all diverge, for all the parameter values we have computed. Therefore, to obtain numerical approximations of such bifurcating steady states, we solve the steady-state system of (5.11) in the form of (2.2) as an initial-value problem with $x$ playing the role of time. This requires information on their pattern because an initial value may neither generate nor converge to such a periodic solution if this solution is not a stable solution of the ODE system. We thus perform a pattern analysis in the following. The result, in addition to the pattern characterization, also provides clear description about the hyperbolic bifurcation regime of the system at 1 : 1 resonance, which will be studied in the next section.

Here we express $P_{ss}(x)$ in the form of (2.4) and substitute into the steady-state system of (5.5). We have

$$r_1\varepsilon''(x) = -\mu_1(a_1r_1\varepsilon(x) + b_1r_1^3\varepsilon^3(x) + c_1r_1^2r_2\varepsilon^3(x)),$$

$$r_2\varepsilon''(x) = -\mu_2(a_2r_2\varepsilon(x) + b_2r_2^3\varepsilon^3(x) + c_2r_1r_2^2\varepsilon^3(x)).$$

Using the product $u_{xx}v_{xx}$ to analyze the relative concavity of the $u$, $v$ components, we find $u_{xx}v_{xx} = \mu_1^2a_1^2r_1r_2 + O(r^4)$ at $x_c$. One sees that this product may have either sign, which indicates that the $P_{ss}(x)$ of this system at 1 : 1 resonance have two different oscillatory patterns. We will show in the following that this is indeed the case. At the points where $\varepsilon''(x) = 0$, the trajectory of $P_{ss}(x)$ penetrates the $u_x$, $v_x$-plane and $\varepsilon(x) = 0$. For those points where $\varepsilon''(x) \neq 0$, we know $r_1r_2 \neq 0$, then we may take the ratio of these two equations and cross multiply, we get

$$\frac{\mu_1}{\mu_2} = \frac{r_1(a_2r_2\varepsilon(x) + b_2r_2^3\varepsilon^3(x) + c_2r_1r_2^2\varepsilon^3(x))}{r_2(a_1r_1\varepsilon(x) + b_1r_1^3\varepsilon^3(x) + c_1r_1^2r_2\varepsilon^3(x))}.$$

At 1 : 1 resonance, $\mu_2 = (a_1/a_2)\,\mu_1$, it is simplified to

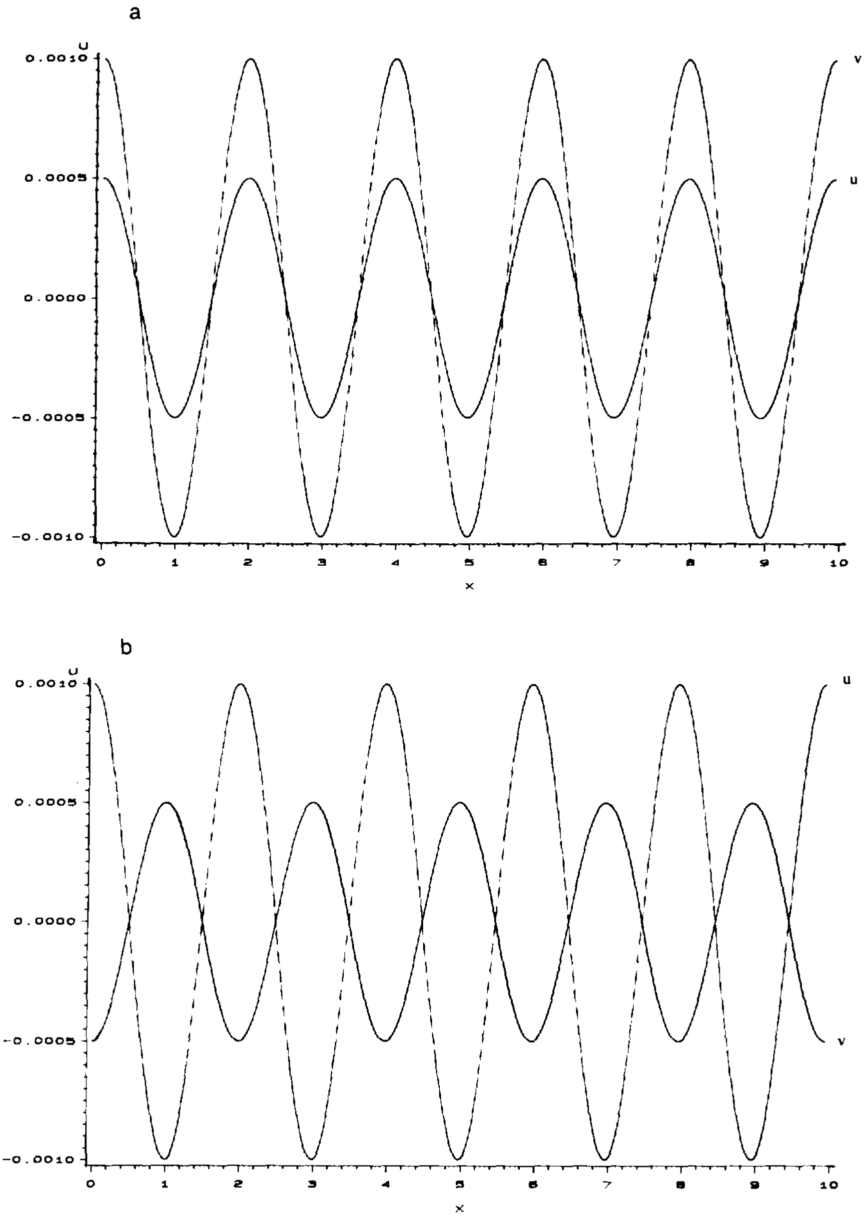
$$a_2b_1r_1^2 - a_1b_2r_2^2 = (a_1c_2 - a_2c_1)r_1r_2. \tag{5.12}$$

This equation determines the pattern of $P_{ss}(x)$ on $[0, l]$.

To illustrate our analysis numerically, we choose $a_1 = 1$, $a_2 = 2$, $b_1 = -1$, $b_2 = -2$, $c_1 = -1$, $c_2 = 1$. In this case we see that the pattern of $P_{ss}(x)$ is described by

$$(2r_1 - r_2)(r_1 + 2r_2) = 0. \tag{5.13}$$

Each of these two factors corresponds to a family of periodic steady states emanated from $Q_k$, and their components have different relative concavity. We can easily choose an initial value $(u, u_x, v, v_x) = (r_1, 0, r_2, 0)$ with $r_1$, $r_2$ satisfying (5.13), which will then generate a periodic solution. The reason to choose initial values with $u_x = v_x = 0$ is to satisfy the no-flux boundary conditions of the RD-system. Presented in Figs. 4a and 4b are symmetric cycles obtained with $(\mu_1, \mu_2) = (10, 5)$ and let the initial conditions be $(u, u_x, v, v_x) = (0.0005, 0, 0.001, 0)$ and $(0.001, 0, -0.0005, 0)$, respectively. These solutions have the same period, an evidence of the 1 : 1 resonance. Observe that each wave of these solutions are not only symmetric with respect to vertical lines through the local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0.0005
0.0000
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-0.0010
0 1 2 3 4 5 6 7 8 9 10
x
u
v

FIGURE 4

extrema (Fix $I_1$), but also symmetric with respect to the horizontal line $u = v = 0$ (Fix $I_2$). In other words, they must satisfy $\max(u(x)) + \min(u(x)) = \max(v(x)) + \min(v(x)) = 0$. In the next section, we will study the hyperbolic bifurcation of the lines intersected by the two families of symmetric cycles and Fix $I_1$ on $r_1$, $r_2$-plane by varying the parameters $(\mu_1, \mu_2)$ near 1 : 1 resonance line.

## 6. Hyperbolic Bifurcation in Predator–Prey Model

In reversible systems, each family of the symmetric cycles emanated from an equilibrium at resonance form a smooth 2-dimensional manifold invariant under the vector field. Parametrized by a variable $\varepsilon$, such a manifold is diffeomorphic to a disc foliated into concentric circles and the period of the symmetric cycles vary smoothly with $\varepsilon$. Moreover, these manifolds are symmetric in the phase space with respect to the set of fixed points of the involution $I$, and they intersect Fix $I$ at 1-dimensional curves. To study the bifurcation of such symmetric cycles, one then tries to characterize the evolution of such curves on Fix $I$ when parameters vary. The steady-state system of model 2 discussed in the previous section is reversible with respect to the two involutions $I_1$ and $I_2$. We will study, in this section, the bifurcation regime at 1 : 1 resonance in this system by investigating the curves intersected by the families of symmetric cycles and Fix $I_1$. All essential information can be obtained through a pattern analysis.

Recall the steady-state system of model 2,

$$\begin{aligned} u_{xx} &= -\mu_1(a_1u + b_1u^3 + c_1u^2v), \\ v_{xx} &= -\mu_2(a_2v + b_2v^3 + c_2uv^2). \end{aligned} \tag{6.1}$$

For $(\mu_1, \mu_2) \in \Gamma_{11}$ (that is, $\mu_2 = (a_1/a_2)\,\mu_1$), the periodic steady states bifurcated from $Q_k = (0, 0)$ can be expressed in the form

$$P_{ss}(x) = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \varepsilon(x), \tag{6.2}$$

with $r_1, r_2 \in R$, $|\varepsilon(x_c)| = 1$, $\varepsilon'(x_c) = 0$, and $\varepsilon(x + T) = \varepsilon(x)$, $T \approx 2\pi/\sqrt{\mu_1 a_1}$. If $(\mu_1, \mu_2) \notin \Gamma_{11}$ but stay near $\Gamma_{11}$, the symmetric cycles near $Q_k = (0, 0)$ can still be approximated by (6.2). Substitute (6.2) into (6.1), one obtains

$$\begin{aligned} r_1\varepsilon''(x) &= -\mu_1(a_1r_1\varepsilon(x) + b_1r_1^3\varepsilon^3(x) + c_1r_1^2r_2\varepsilon^3(x)), \\ r_2\varepsilon''(x) &= -\mu_2(a_2r_2\varepsilon(x) + b_2r_2^3\varepsilon^3(x) + c_2r_1r_2^2\varepsilon^3(x)). \end{aligned} \tag{6.3}$$

Since we are interested in the bifurcation of curves intersected by the manifold of symmetric cycles and Fix $I_1$, that is, where the symmetric cycles attain their local extrema, we consider the division between the two equations in (6.3) at the points $x_c$ where $\varepsilon'(x_c) = 0$. Because of the symmetry of $P_{ss}(x)$ with respect to Fix $I_1$, it is clear that $\varepsilon''(x_c) \neq 0$. Dividing these two equations, cross multiply and simplify, one obtains the following equation describing the curves on Fix $I_1$ intersected by the manifold of symmetric cycles,

$$\mu_1 b_1 r_1^2 + (\mu_1 c_1 - \mu_2 c_2) r_1 r_2 - \mu_2 b_2 r_2^2 + (\mu_1 a_1 - \mu_2 a_2) = 0. \quad (6.4)$$

This equation governs the pattern of symmetric cycles emanated from $Q_k$ at 1 : 1 resonance. Moreover, since $(\mu_1 c_1 - \mu_2 c_2)^2 + 4\mu_1\mu_2 b_1 b_2 > 0$, this equation describes a hyperbola for $(\mu_1, \mu_2)$ near $\Gamma_{11}$, and describes two intersecting lines (Eq. (5.12)) if $(\mu_1, \mu_2) \in \Gamma_{11}$. This means that periodic steady states exist for parameters on both sides of $\Gamma_{11}$, and their pattern are approximated by (6.4).

We use the following parameter values for numerical illustrations: $a_1 = 1$, $a_2 = 2$, $b_1 = -1$, $b_2 = -2$, $c_1 = -1$, $c_2 = 1$. In this case, $\Gamma_{11}$ is given by $\mu_1 = 2\mu_2$ and (6.4) becomes

$$\mu_1 r_1^2 + (\mu_1 + \mu_2) r_1 r_2 - 2\mu_2 r_2^2 + (2\mu_2 - \mu_1) = 0. \quad (6.5)$$

In particular, when $(\mu_1, \mu_2) \in \Gamma_{11}$, (6.5) can be written in factored form

$$(2r_1 - r_2)(r_1 + 2r_2) = 0, \quad (6.6)$$

with the two factors describing the pattern of $P_{ss}(x)$ in each of the two interacting families. In Figs. 5a–5c, we show the graphs of (6.5) for $(\mu_1, \mu_2)$ on and to each side of $\Gamma_{11}$. The values of $(\mu_1, \mu_2)$ in Figs. 5a–5c are (1, 0.6), (1, 0.5), (1, 0.4), respectively. Because of the pattern Eq. (6.6) is exact, it is easy to obtain numerical approximations of $P_{ss}(x)$ at 1 : 1 resonance. Numerical approximations of small-amplitude $P_{ss}(x)$ when $(\mu_1, \mu_2) = (10, 5)$ are obtained and presented in Figs. 4a and 4b. The periods of these solutions are $T_1 = T_2 \approx 1.985$. In Figs. 6a and 6b we present, from each interacting families at 1 : 1 resonance ($(\mu_1, \mu_2) = (10, 5)$), large-amplitude periodic steady states. These solutions are obtained using the following respective initial values (0.5, 0, 1, 0) and (1.4, 0, −0.7, 0). Their periods are $T_1 \approx 3.1167$, $T_2 \approx 5.355$, respectively. One sees that for $P_{ss}(x)$ close to $Q_k$, that is, the small-amplitude solutions, their periods are close to $2\pi/\sqrt{\mu_1 a_1} \approx 1.987$ (Fig. 4a, 4b). For $P_{ss}(x)$ far away from $Q_k$ (the large-amplitude solutions), their periods increase (Figs. 6a, 6b).

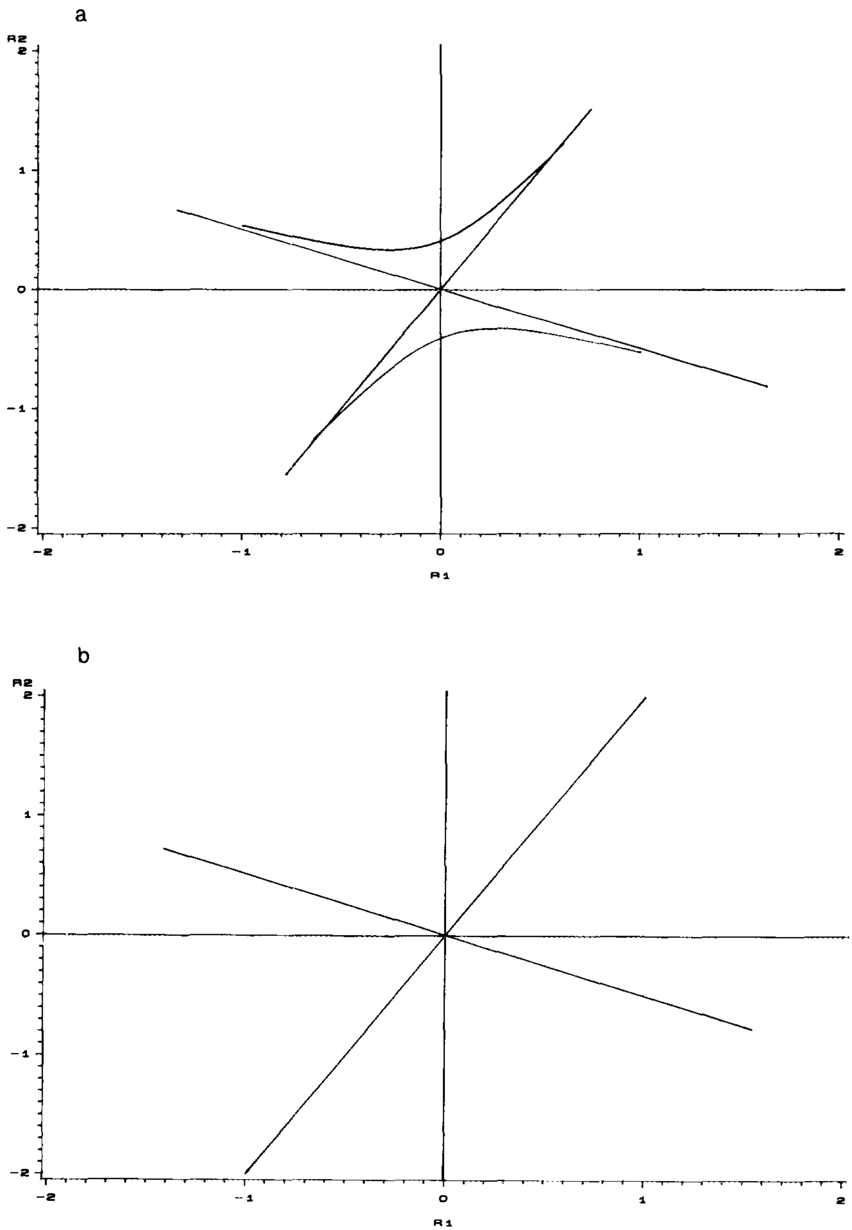


FIGURE 5

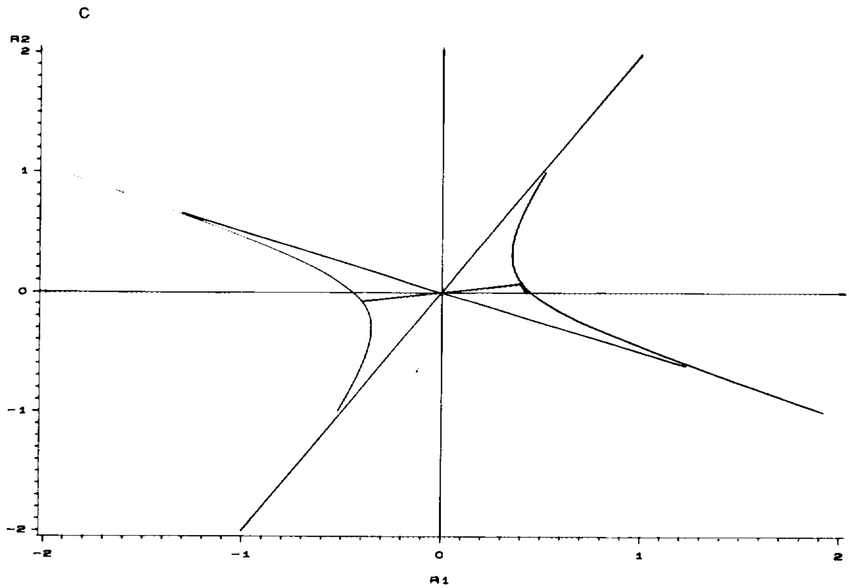


FIG. 5—*Continued*

To approximate the $P_{ss}(x)$ numerically when $(\mu_1, \mu_2)$ is close to $\Gamma_{11}$, one may choose initial values satisfying (6.5). Approximation of a $P_{ss}(x)$ with $(\mu_1, \mu_2) = (1, 0.4)$ is illustrated in Fig. 7a, where the initial values (0.400351, 0, 0.08, 0) are used. For small $x$, this approximation is close. The projection of the trajectory of this solution on the $u$, $v$-plane remains close to the segment connecting the two branches of the hyperbola (a symmetric cycle) shown in Fig. 5c. But for large $x$, the approximation of an initial value that (6.5) provides is not satisfactory (Fig. 7b). This is because that (6.2) is an exact form of the symmetric cycles when $(\mu_1, \mu_2) \in \Gamma_{11}$. For $(\mu_1, \mu_2) \notin \Gamma_{11}$, (6.2) is a reasonable approximation of the symmetric cycles only if $(\mu_1, \mu_2)$ is near $\Gamma_{11}$ and $x$ is small.

## ACKNOWLEDGMENTS

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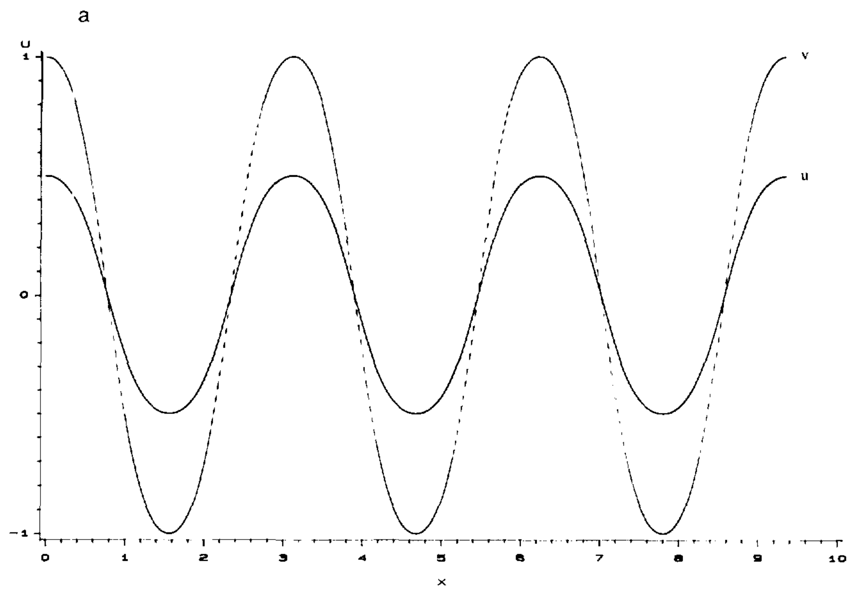


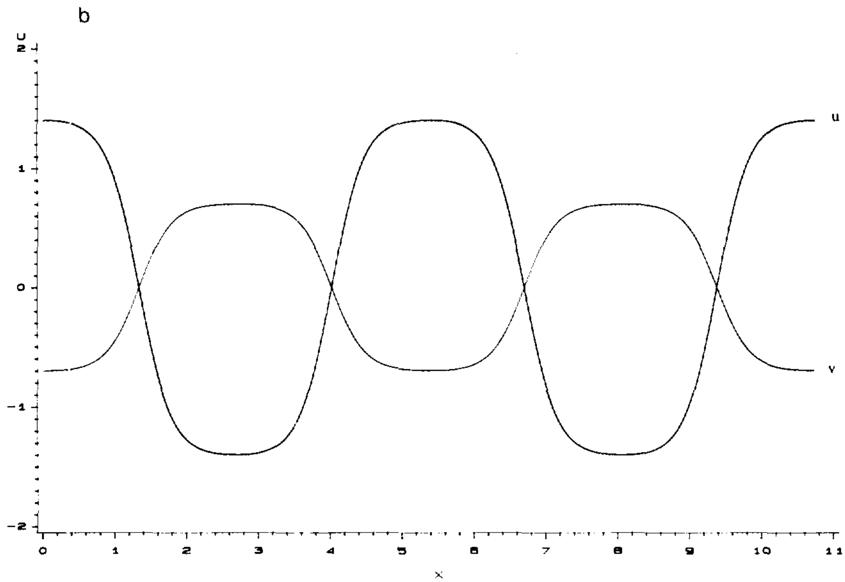


FIGURE 6

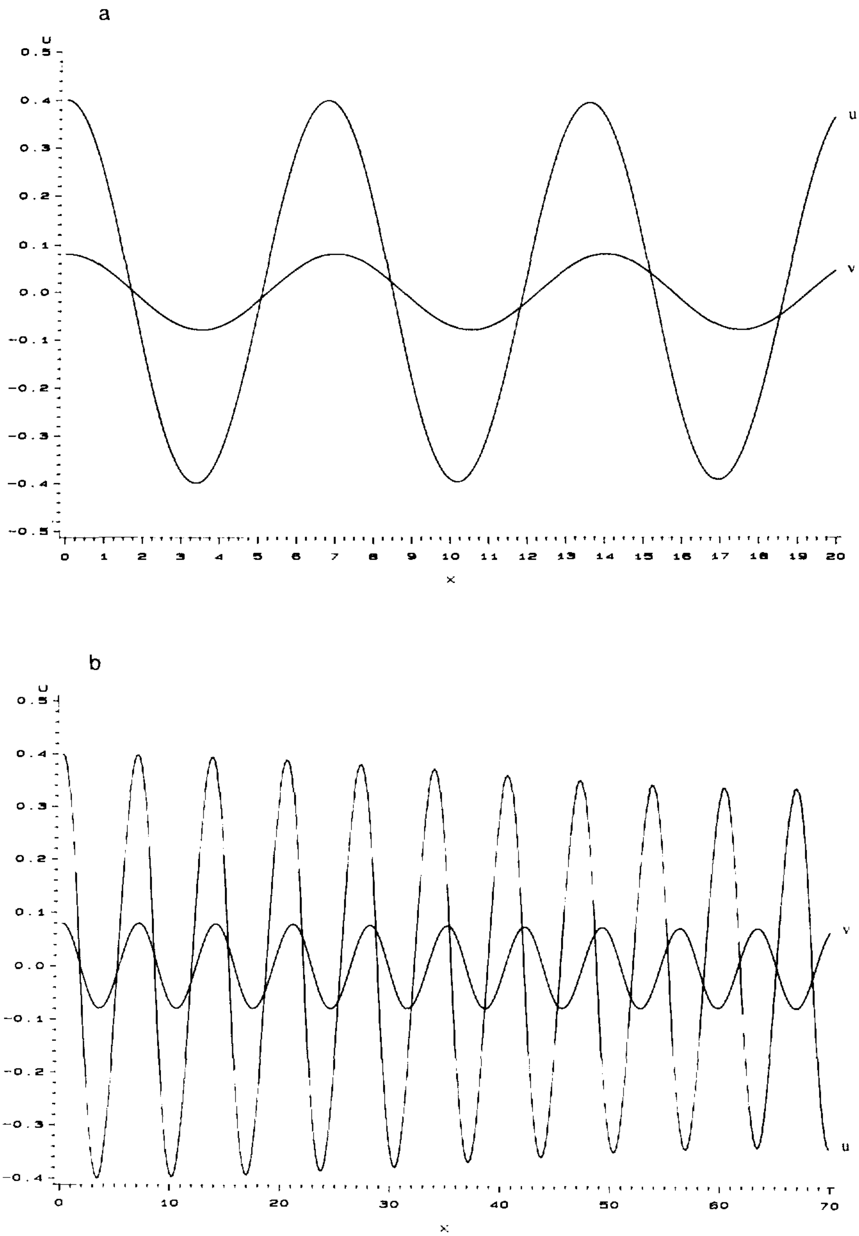
a
u
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
x
u
v
b
u
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
-0.4
0 10 20 30 40 50 60 70
x
v
u

FIGURE 7

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